On the first G_1 stiff fluid spike solution in General Relativity

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THE CONTEXT OF MY TALK

- Application of classical general relativity to the cosmological modeling

- We want to solve the Einstein equations $R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = T_{\mu\nu}$:
  no cosmological constant appearing

- We want to find new exact analytical **inhomogeneous** solutions in cosmology for addressing the **structure formation problem** without the need of invoking the existence of any exotic fluid, or scalar field, or quantum mechanical process, and without the need of restricting ourselves to perturbation theory

- Studying the way the solution approaches the initial time singularity we can check whether we fulfill the **Belinskii-Lifshitz-Khalatnikov locality conjecture**, according to which the spacetime approaches homogeneity and isotropy, implying that spatial derivatives can be neglected in favor of time derivatives inside the Einstein equations which can therefore be reduced to a system of ordinary differential equations (Lifshitz E M and Khalatnikov I M 1963 Adv. Phys. 12 185; Belinskii V A, Khalatnikov I M and Lifschitz E M 1970 Adv. Phys. 19 525; Belinskii V A, Khalatnikov I M and Lifschitz E M 1982 Adv. Phys. 31 639)

APPLICATIONS OF THE GEROCH-STEPHANI GENERATING TECHNIQUE

Theorem. Let g_{ab} be a known exact solution to the EFE for vacuum or stiff matter (for which the energy density and pressure are related by the equation of state $p = \rho$) admitting a Killing vector field (KVF) ξ_a orthogonal to the fluid four-velocity. Then

$$\tilde{g}_{ab} = \frac{\lambda}{\tilde{\lambda}} \left(g_{ab} - \frac{\xi_a \xi_b}{\lambda} \right) + \tilde{\lambda} \eta_a \eta_b \tag{1}$$

is an exact solution to the EFE admitting the KVF ξ_a for vacuum or stiff matter respectively. In the latter case the energy density reads as

$$\rho \qquad \to \qquad \tilde{\rho} = \frac{\rho}{(\cos\theta - \omega \sin\theta)^2 + \lambda^2 \sin^2\theta} . \tag{2}$$

Here we have introduced the following quantities:

$$\frac{\lambda}{\tilde{\lambda}} \quad := \quad \cos^2\theta + (\omega^2 + \lambda^2) \sin^2\theta - 2\omega \sin\theta \cos\theta \tag{3}$$

$$\eta_a \quad := \quad \frac{\xi_a}{\tilde{\lambda}} + 2\cos\theta \sin\theta \alpha_a - \sin^2\theta \beta_a , \tag{4}$$

and solved the following system of partial derivative equations:

$$
\begin{aligned}
\nabla_{[a}\alpha_{b]} &= \frac{1}{2}\epsilon_{abcd}\nabla^c\xi^d && (1)\\
\xi^a\alpha_a &= \omega && (2)\\
\nabla_{[a}\beta_{b]} &= 2\lambda\nabla_a\xi_b + \omega\epsilon_{abcd}\nabla^c\xi^d && (3)\\
\xi^a\beta_a &= \omega^2 + \lambda^2 - 1\,, && (4)
\end{aligned}
$$

with respect to the forms α_a and β_b. Finally $\lambda := \xi_a\xi^a$ and $\omega_a := \epsilon_{abcd}\xi^b\nabla^c\xi^d$, ϵ_{abcd} being the completely antisymmetric Levi-Civita symbol, denote the norm of the KVF and its twist. ω is defined such that $\omega_a = \nabla_a\omega$.

Comments about this formalism:

- Useful for generating inhomogeneous cosmological solutions
- This is a trial and error method: you pick a seed and try to apply it; then you may have to stop because ω is a constant, you cannot integrate the system for β_a in a closed form, the solution you get is still homogeneous, etc...
- The "new" metric you get maybe already known, but written in a funny coordinate system (equivalence problem)

References: Geroch R 1971 J. Math. Phys. 12 918; Geroch R 1972 J. Math. Phys. 13 394; H. Stephani, J. Math. Phys. 29, 1650 (1988); Garfinkle D, Glass E N and Krisch J P 1997 Gen. Relativ. Grav. 29 467

Our original results:

- θ and ω_0 (the constant of integration for the twist) count only as one degree of freedom. Thus we can set $\theta = \frac{\pi}{2}$ and we do not need to integrate the pdes for α_a
- Thus this method actually provides a new one-parameter family of exact solutions in general relativity and not just one solution
- Generation of the first G_1 stiff fluid spiky solution in general relativity (considering a Bianchi V seed)

Meaning of these words:

- G_1: the metric admits one and only one killing vector field, which is the one we used for applying this technique
- **Stiff fluid**: the matter content entering the stress-energy tensor appearing in the Einstein equations obeys to the equation of state $p = \rho$

- **Spike**: our solution is spatially inhomogeneous and in particular it displays a sharp profile in the matter density parameter
- **Seed**: the metric we start from

Seed in our papers:

$$ds^2 = \sinh(2t)[-dt^2 + dx^2 + e^{2x}(\tanh^s(t)dy^2 + \tanh^{-s}(t)dz^2)]$$

Special subcases
(see the book Ed. by J. Wainwright and G. F. R. Ellis, "Dynamical systems in cosmology", Cambridge University Press, Cambridge, 1997):

- $s = 0$: isotropic open Friedmann solution
- $s = \pm 1$: Bianchi V solution for stiff matter
- $s = \pm\sqrt{3}$: vacuum Joseph solution

KVFs of this seed:

$$\begin{aligned} \xi_1 &= \partial_x - y\partial_y - z\partial_z \\ \xi_2 &= \partial_y \\ \xi_3 &= \partial_z \end{aligned}$$

A CLOSE UP LOOK AT OUR NEW SOLUTION

- Our original solution can be written in a closed form in terms of elementary functions

- Obtain it transforming to a new coordinate system such that $t = T$, $x = X$, $y = Ye^{-X}$, $z = Ze^{-X}$ for simplifying to $\xi_1 = \partial_X$

- $\lambda = \frac{\sinh(2T)(Y^2 \tanh^2 T + Z^2 + \tanh(T))}{\tanh(T)}$

- $\omega = 2YZ + \omega_0$

-

$$
\begin{aligned}
\beta_X &= \lambda^2 + \omega^2 - 1 \\
\beta_Y &= -\frac{1}{2} Y \sinh(4T) - 4Y(Y^2 + Z^2)\sinh^4 T + 2YT + 2\omega_0 Z \cosh(2T) \\
\beta_Z &= -\frac{1}{2} Z \sinh(4T) - 4Z(Y^2 + Z^2)\cosh^4 T + 2ZT - 2\omega_0 Y \cosh(2T)
\end{aligned}
$$

- $p = \rho = \frac{1}{4\sinh^3(T)\cosh^3(T)\left[\frac{\sinh^2(2T)(Y^2\tanh^2 T + Z^2 + \tanh(T))^2}{\tanh^2(T)} + (2YZ + \omega_0)^2\right]}$

MATHEMATICAL AND PHYSICAL PROPERTIES OF THE NEW SOLUTION

- This new solution is of the **most general Petrov type I**.

  In fact the asymptotic form of the early-time attractor, which is a Jacobs stiff fluid solution, is of Petrov type I, and hence the new solution must also be of Petrov type I.

- Late time behavior of some invariants:

  Hubble function: $H = \frac{10\sqrt{2}}{3(Y^2+Z^2+1)} e^{-3T} + O(e^{-5T})$

  Matter parameter: $\Omega_m = \frac{24}{25} e^{-4T} + O(e^{-6T})$

  Curvature parameter: $\Omega_k = \frac{9}{25} + O(e^{-2T})$

  Shear: $\Sigma^2 = \frac{16}{25} + O(e^{-2T})$

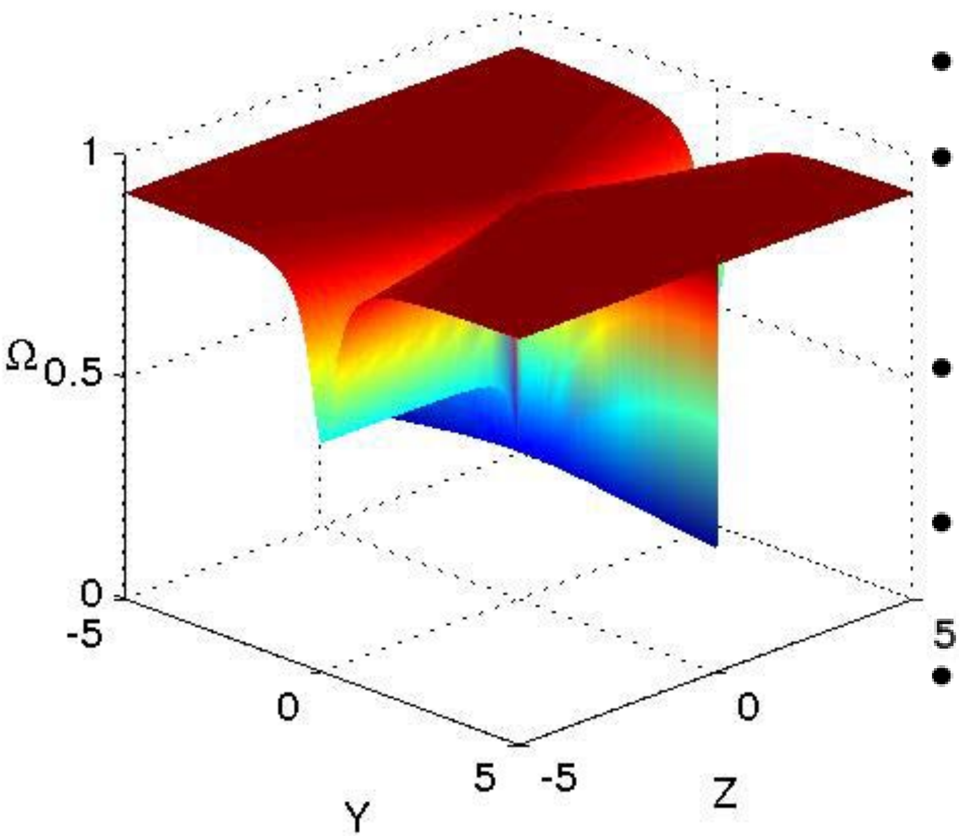
- Our solution **anisotropically** approaches a **vacuum** state. Indeed, the transformed solution does not isotropize since Σ^2 asymptotes to a non-zero constant value at late times.

  Moreover the asymptotic late time vacuum state is **not Milne** since its Weyl tensor is non-zero.

THE SPIKY BEHAVIOR

- Mathematically spikes can be located looking at the inhomogeneity parameter $\frac{\omega^2-\lambda^2}{\omega^2+\lambda^2}$. Usually a spike occurs at the place where $\omega \to 0$ as λ becomes small.

- If $\omega_0 = 0$ spikes form along the planes $Y = 0$ and $Z = 0$, with a spike intersection on the line $Y = Z = 0$



- Crossing spikes in the matter parameter
- Persistence of the spike towards the big bang: counterexample to the BKL locality conjecture
- Spikes are possible seeds for the structure formation problem
- Possible explanation for the astrophysical voids and walls
- All these points are due to the non linearities in the Einstein equations and we do not need any exotic field or matter

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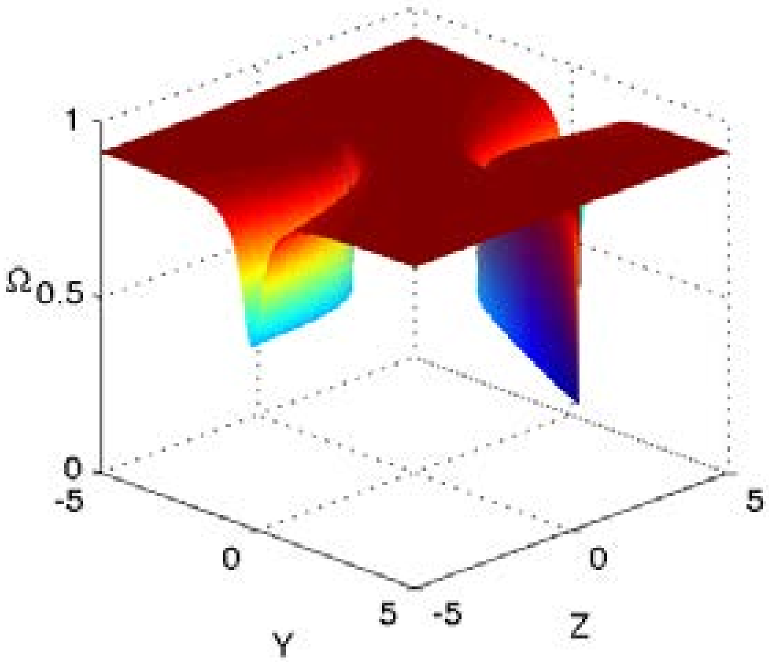
MORE ON THE SPIKY BEHAVIOR

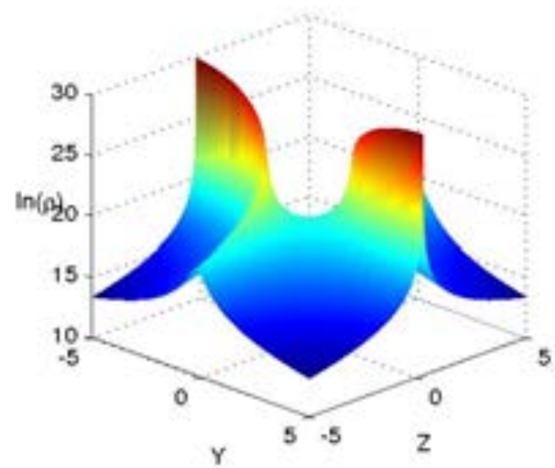
- As $T \to 0$ (towards the singularity) and $\omega_0 = 0$ the inhomogeneity parameter behaves as $\frac{\omega^2-\lambda^2}{\omega^2+\lambda^2} \to \frac{Y^2-Z^2}{Y^2+Z^2}$ which is discontinuous on the intersection $Y = 0 = Z$. Thus these spikes persist at the singularity

- It can be shown that these spikes become narrowest at the singularity and instead disappear at late times

- If $\omega_0 \neq 0$ then spikes are still present, but do not intersect. Also in this case they disappear at late times.

- The figures display the snapshot of the spatial profile of the matter parameter and of the energy density at time $T = 0$





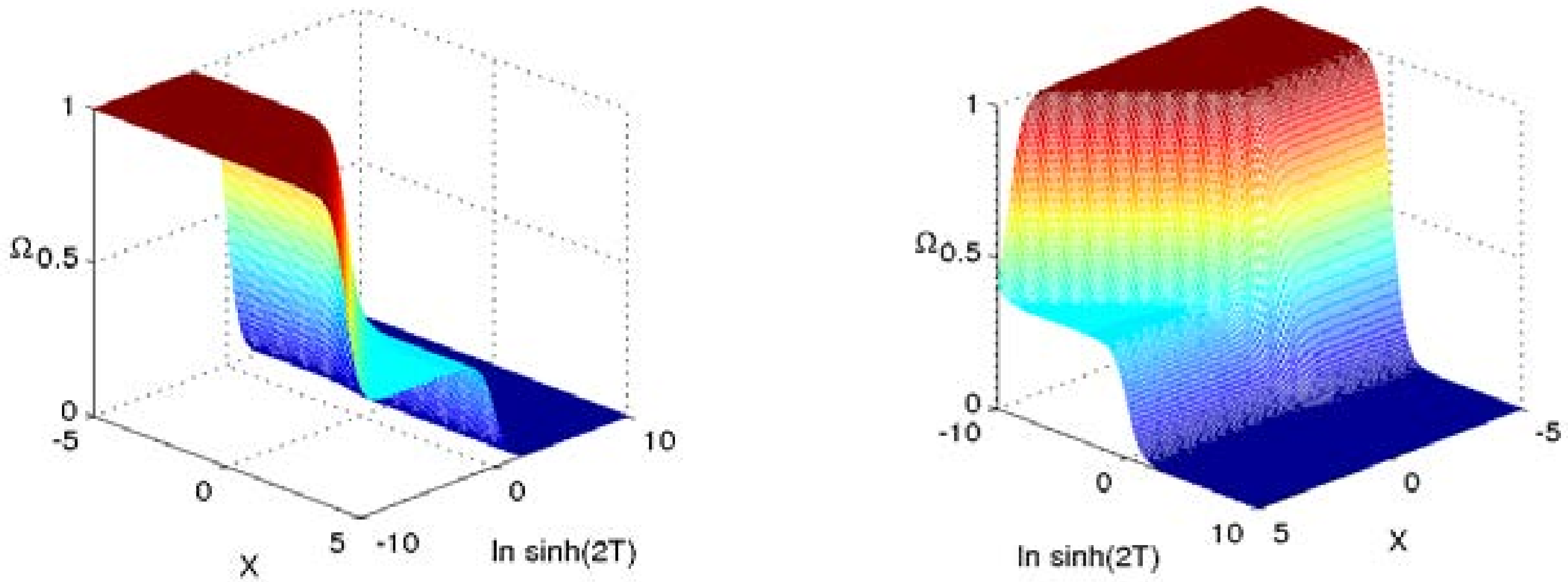
CLOSE TO FL SOLUTIONS

- Here we consider a Friedmann metric as a seed, i.e.

$$ds^2 = \sinh(2t)[-dt^2 + dx^2 + e^{2x}(\tanh^s(t)dy^2 + \tanh^{-s}(t)dz^2)]$$

  with $s = 0$ and KVF $\partial_y + \partial_z$

- The figures show that Ω tends to 1 at the bang time, indicating that the solution is past asymptotic to the FL solution, i.e. that is closed to isotropy (as from the Gauss constraint)



CONCLUDING REMARKS

- Astrophysical observations suggest that the Universe is dominated by bubbles of large voids surrounded by denser walls
- It has also been argued that some of the large scale observational anomalies may require the existence of non-linear structures in the early Universe
- For example, most galactic nuclei contain supermassive black holes which must have already existed by a redshift of $\sim$10. However, such enormous black holes could not have naturally formed so early within the conventional cosmological model, and there must have already existed large seed black holes well before galaxy formation (with these seeds subsequently growing through accretion)
- Intersecting spikes may present an interesting model for the formation of such non-linear structures in general relativity
- In future works we want to find solutions exhibiting spikes at late times
- Solutions close to FL may be of interest for researchers working on perturbation theory who require an exact solution to compare against